

## On Phase Space and Statistics of Continua

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Problems in introducing suitable phase space and statistics occur for continua and degenerate discrete systems. The solution of these problems for the Korteweg-de Vries equation is discussed. The classical removal of the ultraviolet catastrophe in this case is contrasted with Planck's black-body radiation spectrum.

The statistics of continua is basic in several areas of physics and especially in quantum theory and turbulence. Statistics of continua far from equilibrium necessitates the solution of equations [1] with functional derivatives for which no mathematical tools exist. Thermodynamics of continua is, however, an important and challenging discipline and lies on the border of the mathematically tractable, so that this paper is restricted to equilibrium statistics of conservative continuous systems.

Conventional Gibbs statistics for systems of interacting particles starts with a canonical phase space  $p_i, q_i$  with  $i = 1, \dots, N$ . From the Hamiltonian dynamics

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad (1)$$

one can prove conservation of the volume element (Liouville theorem)  $d\Omega$ :

$$\frac{d}{dt}(d\Omega) = \frac{d}{dt} \left( \prod_{i=1}^N dp_i dq_i \right) = 0. \quad (2)$$

This, combined with an ergodic or "chaos" property leads to microcanonical and canonical Gibbs distributions [2]. The canonical distribution is

$$P = e^{-\beta H} / Z, \quad (3)$$

where

$$Z = \int e^{-\beta H} \prod_{i=1}^N dp_i dq_i \quad (4)$$

is the partition function.

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If the  $p_i, q_i$  are mapped one to one into a new set of noncanonical variables  $z_j$  with  $j = 1, \dots, 2N$ , Eqs. (1) become [3]

$$\dot{z}_j = \eta_{jk} \frac{\partial H}{\partial z_k}, \quad (5)$$

and

$$d\Omega = \sqrt{|\eta_{jk}^{-1}|} \prod_{i=1}^{2N} dz_i. \quad (6)$$

The main change in the canonical distribution and the partition function will be made to the weighting Jacobian, so that

$$Z = \int e^{-\beta H} \sqrt{|\eta_{jk}^{-1}|} \prod_{i=1}^{2N} dz_i. \quad (7)$$

For general systems, (5) remains valid but the antisymmetric  $M \times M$  matrix  $\eta_{jk}$  can be singular [3] as for the rigid top (the rigid top is a kind of degenerate continuum). Such systems possess Casimir invariants [3, 4] which commute with all observables. Their number  $m$  is equal to the degree of degeneracy of the matrix  $\eta_{jk}$ . For each set of Casimirs  $C_i$  one can obtain [4] a symplectic system with a nonsingular matrix  $\eta'_{ij}$  of even dimension  $M-m$ . In this case the canonical distribution does not seem to be uniquely defined:

$$P = \exp \left\{ -\beta_0 H'(z_i, C_j) - \sum_{j=1}^m \beta_j f_j(C_k) \right\} / Z, \quad (8)$$

with  $i = 1, \dots, M-m, j = 1, \dots, m, k = 1, \dots, m$ , and

$$Z = \int \exp \left\{ -\beta_0 H'(z_i, C_j) - \sum_{j=1}^m \beta_j f_j(C_k) \right\} \cdot \sqrt{|\eta'_{ij}{}^{-1}|} \prod_{i=1}^{M-m} dz_i \prod_{j=1}^m dC_j. \quad (9)$$

The freedom in the functions  $f_j$  can only be removed by more information about the system, e.g. by prescribing specific values for the  $C_j$ ,  $\beta_j = 0$  and carrying the integration over the  $z_i$  only.

In the case of fluids and plasmas the matrix  $\eta_{ij}$  is replaced [5, 6] by differential or integral operators  $\mathbf{A}$ , and the  $z_j$  by functions  $\mathbf{u}$ , so that the system can be written noncanonically as

$$\dot{\mathbf{u}} = \mathbf{A} \frac{\partial H}{\partial \mathbf{u}}. \quad (10)$$

Generalized Poisson brackets in Eulerian variables can be constructed in the form of Lie-Poisson

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brackets [6]. These brackets are in general [4] singular and possess Casimir invariants. The difference to the discrete singular case is that it is very difficult to find the whole set of Casimir invariants and the corresponding [4] transformation of variables which reduces the system to a symplectic system of lower dimension. This problem is in fact not solved, so that it is not possible in general to define a volume element and do statistics in Euler variables.

If we go to Clebsch variables, we obtain a canonical system at the cost of inflating phase space. This means that we know how to do statistics in a non-physical space, but we do not know how to fix the "gauge" freedom and translate the results back [7] to physical space. With nonlinear Lagrange variables one can, in principle, introduce a canonical formalism but then the expressions are tedious and the calculations virtually impossible. There is also the question of fixing the labelling freedom. In linearized Lagrange variables all this can be done and the results were given in [8]. Linearization leads, however, to a Gaussian canonical distribution which itself leads to equipartition of energy and the ultraviolet catastrophe.

A special case for which a solution to all these difficulties has been found [9] is the Korteweg-de Vries (KdV) equation

$$u_t - 6uu_x + u_{xxx} = 0. \quad (11)$$

It can be transformed [10] to a "quasi"-canonical form by

$$u = v^2 + v_x. \quad (12)$$

One obtains the modified KdV equation

$$v_t - 6v^2v_x + v_{xxx} = 0 \quad (13)$$

or

$$v_t = \frac{\partial}{\partial x} \frac{\delta H}{\delta v}, \quad (14)$$

with

$$H = \frac{1}{2} \int (v^4 + v_x^2) dx. \quad (15)$$

Volume element and Statistics in  $v$  space are possible. Miura transformation (12) seems similar to a Clebsch decomposition, but in contrast to Clebsch potentials it deflates phase space [10]. The problem is not in restricting the "gauge" freedom but in understanding what has been lost. Partial answers [11] exist for the case  $-\infty < x < +\infty$ . It is proved that solutions asymptotically exhibiting layers are lost. It is not known whether periodic solutions are lost by Miura transformation. I would conjecture a faithful phase space transformation in the case of the periodic solutions needed in [9] and [12]. Let us finally mention that without any transformation of phase space, equations of the type

$$u_t - u^{2n}u_x + u_{xxx} = 0 \quad (16)$$

have a simple and divergence-free statistical treatment using calculations similar to those of [9] and [12], but they are not physically appealing.

One of the fascinating results is that the  $k$ -spectrum obtained [12] for KdV is of the Lorentz type and is free of divergences. Besides possible applications in several areas of physics, the removal of the ultraviolet catastrophe is interesting in itself because it happens in a purely classical way without need of quantisation, unlike in the case of Planck's black-body radiation. The point with Maxwell's equations is that they are linear and valid up to the highest energies attained hitherto, i.e. energies much higher than in black-body radiation, and there is no sign that they will have to be modified in the future. This means that there cannot be [13] a classical interpretation for black-body radiation because Maxwell's equations are linear and cannot contain sizeable dispersive terms like KdV or (16).

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